ARROW'S THEOREM BY BOOLEAN FOURIER ANALYSIS

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1. Introduction and Review

Nearly everything in these notes was scraped from [9] and reformatted to fit into a single self-contained lecture. We work with boolean functions

$$f : \{-1, 1\}^n \to \{-1, 1\}.$$

Note that we are using -1 and 1 instead of 0 and 1 as would usually be expected when using the term "boolean." The theory is the same but using $\{-1, 1\}$ instead of $\{0, 1\}$ will be notationally convenient for us. Elements $x \in \{-1, 1\}^n$ will be interchangeably thought of as subsets of $[n]$ through the obvious bijection $x \longmapsto \{i \in [n] : x_i = 1\}$.

We then have the usual L^2 inner product, for $f, g : \{-1, 1\}^n \to \mathbb{R}$ we set

$$\langle f, g\rangle := \mathbb{E}[fg] = \sum_{x\in\{-1,1\}^n} \frac{1}{2^n} f(x)g(x).$$

We may then define the *Fourier transform* $\widehat{f} : \{-1, 1\}^n \to \mathbb{R}$ of f by expressing f in a new orthonormal *Fourier basis* $\{\chi_S\}_{S\subseteq[n]}$ and letting $\widehat{f}(S) := \mathbb{E}[f\chi_S]$ be the coefficient of χ_S in f's Fourier basis expansion. The basis elements are given by

$$\chi_S(x) := \prod_{i\in S} x_i = \begin{cases} +1 & \text{an odd number of } S\text{-bits of } x \text{ are } -1 \\ -1 & \text{an even number of } S\text{-bits of } x \text{ are } -1 \end{cases}$$

for each $S \subseteq [n], x \in \{-1, 1\}^n$, and thus we have

$$f(x) = \sum_{S\subseteq[n]} \widehat{f}(S)\chi_S(x).$$

The map $\mathcal{F} : f \longmapsto \widehat{f}$ is called the *Fourier transform*. We then have

$$\mathbb{E}[fg] = \sum_{S\subseteq[n]} \widehat{f}(S)\widehat{g}(S)$$

which in particular implies Parseval's identity

$$\|f\|_2^2 = \sum_{S\subseteq[n]} \widehat{f}(S)^2.$$

In our analysis, grouping Fourier coefficients by the number of elements in their underlying sets will be useful, so we define the *Fourier weight* of f at level k to be

$$\mathcal{W}_k(f) := \sum_{|S|=k} \widehat{f}(S)^2.$$

2. Arrow's Theorem

Now let's use boolean functions to talk about elections. The following was originally studied by Condorcet [2] and takes after his name. A 3-candidate Condorcet election amongst n voters with choice function $f : \{-1,1\}^n \to \{-1,1\}$ is held in the following manner. There are three candidates: A, B, C and society needs to rank them with a strict total order $<$. We assume that voters have individual preferences that rank A, B, C in a strict total order, and for each pair of candidates we use the choice function f to take collected votes and choose a winner. The ith bit of input to f corresponds to whether the ith voter chose the first or second candidate in a pairwise election. If v_1, v_2, v_3 are the votes amongst $(A, B), (B, C), (C, A)$, then society's preferences are taken to be $(f(v_1), f(v_2), f(v_3))$.

Unfortunately, even if all voters have strict total orderings on the candidates, the aggregate preference does not necessarily define a strict total order on the candidates. For instance, with majority rule amongst 3 people, if the individual preferences are $(1, -1, 1), (1, 1-1), (-1, 1, 1)$ corresponding to $C > A > B, A > B > C, B > C > A$, then each of $A > B, B > C, C > A$ occurs in 2/3 of the elections, so the societal preferences will be $(1, 1, 1)$ which corresponds to $A > B > C > A$, an irrational outcome.

We say that a triple (x, y, z) in $\{-1,1\}^3$ is *rational* or denotes a *rational outcome* if it defines a strict total order. There are 6 such rational preferences in a 3 candidate election:

$$(1, 1, -1), (1, -1, 1), (1, -1, -1), (-1, 1, 1), (-1, 1, -1), (-1, -1, 1)$$

corresponding to the 6 possible strict total orders on 3 elements. There are two irrational preferences:

$$(1, 1, 1), (-1, -1, -1).$$

Any choice function $f : \{-1,1\}^n \to \{-1,1\}$ can be used for an election. Some choices include:

- $f(x) = 1$, ignore the votes and always pick the first candidate as the winner. Note: this always leads to cyclic preferences.
- $f(x) = \mathrm{Maj}_n(x) = \mathrm{sign}(\sum_{i=1}^n x_i)$, majority rule. We take the sign to be $+1$ in case the sum is zero, just so we can define Maj_n for all n.
- $f(x) = x_i$, a dictator's choice. Disregard all votes except for the ith voter.

In reality, we would like our f to satisfy some properties that we think a fair voting system should have:

- *Unanimity*, $f(11\cdots1) = 1, f(-1-1\cdots-1) = -1$, i.e. if the vote is unanimous, then the winner is the person everyone voted for.
- *Rationality*, $(f(x), f(y), f(z))$ is always a rational choice that defines a strict total order on the candidates.
- *Non-dictatorship*, f is not a dictator's choice.

Since irrational outcomes fail to solve the problem of deciding on a societal ranking amongst A, B, C, we might consider finding voting choice functions that minimize irrationality. Stunningly, Arrow showed [1] that this may not be such a good idea.

Theorem 2.1. *(Arrow's Impossibility Theorem) Let the choice function $f : \{-1,1\}^n \to \{-1,1\}$ be used in a 3-candidate Condorcet election. If f makes rational decisions and respects unanimity, then f is a dictator's choice.*

Proof. (Originally due to [7].) Let $R : \{-1,1\}^3 \to \{0,1\}$ by setting $R(a, b, c)$ to be the indicator of (a, b, c) being a rational choice. Let's compute $\widehat{R}$. Since R is symmetric in all variables, we only have to check $\widehat{R}(\emptyset), \widehat{R}(\{1\}), \widehat{R}(\{1, 2\})$ and $\widehat{R}(\{1, 2, 3\})$. Of course $\widehat{R}(\emptyset) = 6/8 = 3/4$ is the probability that a uniformly chosen choice is rational. Then $\widehat{R}(\{1, 2\}) = \frac{1}{8}(0 - 1 - 1 + 1 + 1 - 1 - 1 + 0) = -\frac{1}{4}$ sums over rational choices $+1$ if an even number of $\{1, 2\}$ are included and -1 if an odd number are included. In a similar manner, we find $\widehat{R}(\{1\}) = \widehat{R}(\{1, 2, 3\}) = 0$. Thus

$$R(a, b, c) = \frac{3}{4} - \frac{1}{4}ab - \frac{1}{4}ac - \frac{1}{4}bc.$$

With this in mind, let's hold a random election. Let $(\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i)$ for $i = 1, \ldots, n$ be iid uniformly drawn among rational choices. Then by assumption we have

$$1 = \mathbb{E}[R(f(\boldsymbol{x}), f(\boldsymbol{y}), f(\boldsymbol{z}))] = \frac{3}{4} - \frac{1}{4}\mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y}) + f(\boldsymbol{x})f(\boldsymbol{z}) + f(\boldsymbol{y})f(\boldsymbol{z})]$$

and since $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}, \boldsymbol{z}), (\boldsymbol{y}, \boldsymbol{z})$ all have the same distribution, using this we have

$$1 = \mathbb{E}[R(f(\boldsymbol{x}), f(\boldsymbol{y}), f(\boldsymbol{z}))] = \frac{3}{4} - \frac{3}{4}\mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y})]$$

so that rearranging gives

$$-\frac{1}{3} = \mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y})].$$

Let's expand both f's in their Fourier basis:

$$\mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y})] = \sum_{S,T\subseteq[n]} \widehat{f}(S)\widehat{f}(T)\mathbb{E}[\chi_S(\boldsymbol{x})\chi_T(\boldsymbol{y})].$$

Since $\chi_S(a) = \prod_{i\in S} a_i$, if $S \neq T$ then there will be a lone $\boldsymbol{x}_i$ or $\boldsymbol{y}_i$ independent of the rest with mean zero that will kill that term. Thus

$$\mathbb{E}[f(\boldsymbol{x})f(\boldsymbol{y})] = \sum_{S\subseteq[n]} \widehat{f}(S)^2\mathbb{E}[\chi_S(\boldsymbol{x})\chi_S(\boldsymbol{y})] = \sum_{S\subseteq[n]} \widehat{f}(S)^2 \prod_{i\in S} \mathbb{E}[\boldsymbol{x}_i\boldsymbol{y}_i] = \sum_{S\subseteq[n]} \widehat{f}(S)^2 \prod_{i\in S}(\mathbb{P}(\boldsymbol{x}_i = \boldsymbol{y}_i) - \mathbb{P}(\boldsymbol{x}_i \neq \boldsymbol{y}_i))$$

$$= \sum_{S\subseteq[n]} \widehat{f}(S)^2 \prod_{i\in S}(2/6 - 4/6) = \sum_{S\subseteq[n]} \left(\frac{-1}{3}\right)^{|S|} \widehat{f}(S)^2 = \sum_{k=0}^{n} \left(\frac{-1}{3}\right)^k \mathcal{W}_k(f).$$

Thus

$$-\frac{1}{3} = \sum_{k=0}^{n} \left(\frac{-1}{3}\right)^k \mathcal{W}_k(f).$$

Now we know that $\sum_{k=0}^{n} \mathcal{W}_k(f) = \sum_{S\subseteq[n]} \widehat{f}(S)^2 = 1$ by Parseval since $f^2 = 1$. Thus, the right hand side of the previous equation is a convex combination of powers of $-\frac{1}{3}$ that adds up to $-\frac{1}{3}$. Since $-\frac{1}{3}$ is also the minimum of the coefficients we are taking a convex combination of, it follows that the combination must be trivial with all weight on this coefficient, i.e. $\mathcal{W}_1(f) = 1$ and $\mathcal{W}_k(f) = 0$ for all $k \neq 1$. Assume for sake of contradiction that $0 < \widehat{f}(\{i\})^2 < 1$ for some i. Then $\widehat{f}(\{i\})^2 < 1$ for all i, so that $\widehat{f}(\{i\})^2 < \left|\widehat{f}(\{i\})\right|$ if $\widehat{f}(\{i\}) \neq 0$. Since some $\widehat{f}(\{i\})$ is nonzero this gives us

$$1 = \sum_{i=1}^{n} \widehat{f}(\{i\})^2 < \sum_{i=1}^{n} \left|\widehat{f}(\{i\})\right| = \sum_{i=1}^{n} \widehat{f}(\{i\}) \operatorname{sign}(\widehat{f}(\{i\})) = f(x^*)$$

for $x_i^* := \operatorname{sign}(\widehat{f}(\{i\}))$ for each i (sign denotes the right continuous sign function; in our case the sign taken at 0 is irrelevant), contradicting the fact that f takes values in $\{-1, 1\}$. It follows that $\widehat{f}(\{i\})^2 \in \{0, 1\}$ for all i so that we may choose some i_0 for which $\widehat{f}(\{i_0\}) \in \{-1, 1\}$, and $\widehat{f}(\{i\}) = 0$ for all $i \neq i_0$. Thus the Fourier expansion of f is $f(x) = \widehat{f}(\{i_0\})x_i$. The unanimity assumption then implies $\widehat{f}(\{i_0\}) = +1$, so that $f(x) = x_{i_0}$ is a dictator's choice. □

Arrow's theorem shows us that the only voting system that makes rational decisions and respects unanimity is a dictatorship. However, perhaps this is just a fluke that occurs because we have restricted ourselves to *perfectly* rational voting systems. What if we allow a small number of irrational outcomes? It turns out that the closer to a rational voting system we have, the closer to a dictatorship or anti-dictatorship the system must be. We can easily extend the proof of Arrow's theorem to the following

Proposition 2.2. *Let the choice function $f : \{-1,1\}^n \to \{-1,1\}$ be used in a 3-candidate Condorcet election. If f makes irrational decisions on at most ϵ percent of possible votes, then $\mathcal{W}_1(f) \geqslant 1 - \frac{9}{2}\epsilon$.*

Proof. Using the notation of the proof of Arrow's theorem, the hypothesis now says $1-\epsilon \leqslant \mathbb{E}[R(f(\boldsymbol{x}), f(\boldsymbol{y}), f(\boldsymbol{z}))]$. We repeat the proof of Arrow's theorem with minor modifications to keep track of the ϵ until we find

$$-\frac{1}{3} + \frac{4}{3}\epsilon \geqslant \sum_{k=0}^{n} \left(\frac{-1}{3}\right)^k \mathcal{W}_k(f).$$

For a given $\mathcal{W}_1(f)$ we can bound the rest of the sum by putting all the remaining mass on $\mathcal{W}_3(f)$, giving

$$-\frac{1}{3} + \frac{4}{3}\epsilon \geqslant -\frac{1}{3}\mathcal{W}_1(f) - \frac{1}{27}(1 - \mathcal{W}_1(f))$$

or equivalently $\mathcal{W}_1(f) \geqslant 1 - \frac{9}{2}\epsilon$. □

Intuitively this says that f is mostly determined by it's singleton Fourier coefficients. We can formalize this by showing it actually means that f is close to being a dictator's choice or the negative of a dictator's choice.

Theorem 2.3. *(Freidgut-Kalai-Naor [4]) Let the choice function $f : \{-1,1\}^n \to \{-1,1\}$ be used in a 3-candidate Condorcet election. If $\mathcal{W}_1(f) \geqslant 1 - \epsilon$, then there is $g : \{-1,1\}^n \to \{-1,1\}$ such that either g or $-g$ is a dictator's choice, and $g \neq f$ on at most $O(\epsilon)$ percent of possible votes.*

In the real world, a majority vote $\mathrm{Maj}_n(x) := \mathrm{sign}\left(\sum_{i=1}^n x_i\right)$ is very commonly used. Since Arrow's theorem or the FKN theorem tells us that only dictatorships can make rational choices, and we know majority rule is nowhere near a dictatorship, how irrational *is* the poster child of democracy? This question was asked by Guilbaud [6] and first answered by Garman and Kamien [5].

Theorem 2.4. *In a 3-candidate Condorcet election, using the majority vote Maj_n, the proportion of rational outcomes approaches*

$$\frac{3}{4} - \frac{3}{4}\left(1 - \frac{2}{\pi}\arccos(-1/3)\right) = \frac{3\arccos(-1/3)}{2\pi} \approx .912$$

as $n \to \infty$. We call the limit "Guilbaud's number."

Proof. As usual we hold a random election. Using the notation of the proof of Arrow's theorem, the proportion of rational outcomes is

$$\mathbb{E}[R(\mathrm{Maj}_n(\boldsymbol{x}), \mathrm{Maj}_n(\boldsymbol{y}), \mathrm{Maj}_n(\boldsymbol{z}))] = \frac{3}{4} - \frac{3}{4}\mathbb{E}[\mathrm{Maj}_n(\boldsymbol{x})\mathrm{Maj}_n(\boldsymbol{y})]$$

so it suffices to calculate the expectation on the right. We have

$$\mathbb{E}[\mathrm{Maj}_n(\boldsymbol{x})\mathrm{Maj}_n(\boldsymbol{y})] = \mathbb{E}\left[\mathrm{sign}\left(\sum_{i=1}^n \boldsymbol{x}_i\right)\mathrm{sign}\left(\sum_{i=1}^n \boldsymbol{y}_i\right)\right] = \mathbb{E}\left[\mathrm{sign}\left(\sum_{i=1}^n \frac{\boldsymbol{x}_i}{\sqrt{n}}\right)\mathrm{sign}\left(\sum_{i=1}^n \frac{\boldsymbol{y}_i}{\sqrt{n}}\right)\right].$$

By the central limit theorem, $\left(\sum_{i=1}^n \frac{\boldsymbol{x}_i}{\sqrt{n}}, \sum_{i=1}^n \frac{\boldsymbol{y}_i}{\sqrt{n}}\right)$ converges in distribution to a bivariate normal (X, Y) with $\mathbb{E}[X] = \mathbb{E}[Y] = 0, \mathbb{E}[X^2] = \mathbb{E}[Y^2] = 1$ and $\mathrm{Cov}(X,Y) = \mathbb{E}[\boldsymbol{x}_1\boldsymbol{y}_1] = -\frac{1}{3}$. In particular

$$\mathbb{E}[\mathrm{Maj}_n(\boldsymbol{x})\mathrm{Maj}_n(\boldsymbol{y})] \to \mathbb{E}[\mathrm{sign}(X)\,\mathrm{sign}(Y)] = 1 - 2\mathbb{P}(\mathrm{sign}(X) \neq \mathrm{sign}(Y)).$$

The remaining work is to show $\mathbb{P}(\mathrm{sign}(X) \neq \mathrm{sign}(Y)) = \frac{1}{\pi}\arccos(-1/3)$. In general replacing $-\frac{1}{3}$ with $\delta \in [-1,1]$ we can note that $Y = \delta X + \sqrt{1-\delta^2}Z$ where $Z \perp X, Y$ is standard normal. The result then follows from easy calculations. □

While .912 percent of outcomes being rational might seem alright for a single election, the ubiquitous use of majority rule implies that there is a substantial amount of irrational vote aggregation going on in the world.

3. References

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