Coupling and Unimodularity

in Stationary Settings

James Thomas Murphy III
Advisor: François Baccelli

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Outline

This thesis is primarily comprised of 3 papers about:

(i) Exact Coupling of Random Walks,

(ii) Doeblin Trees,

(iii) Point-shifts of Point Processes on Groups.

This talk will cover the first two.

Exact Coupling

Definition (Thorisson [3], 2000.)

A **successful exact coupling** is a triple $((S_n)_n, (S'_n)_n, T)$ of two processes and an a.s. finite random time such that

$$S_n = S'_n, \qquad n \geqslant T.$$

Why do this? If there is a successful exact coupling, then

$$\left\| \mathbf{P}(S_n \in \cdot) - \mathbf{P}(S'_n \in \cdot) \right\|_{\mathrm{TV}} \to 0, \qquad n \to \infty.$$

Exact Coupling - History

When is successful exact coupling possible?
Focus on Markov chain case, $S = S^x$ and $S' = S^y$ started at different states x, y.

- Doeblin 1938: For ergodic chain on finite space, coupling always possible.
- Ornstein 1968: For RW on $\mathbb{Z}$, coupling works with aperiodic step-lengths on $\mathbb{Z}$.
- Berbee 1979: For RW on $\mathbb{R}$, coupling is possible if n-step lengths have Lebesgue part for some $n \geqslant 1$.
- Arnaldsson 2010: For RW on $\mathbb{R}$ with countable set A of possible jump sizes, coupling is possible iff $x - y \in \langle A - A \rangle$.
- Thorisson 2011: Thorisson asked for someone to fill in the gap between last two cases.

Exact Coupling - Setup

Let G be an Abelian Polish group.

Definition (Thorisson [3], 2000.)

For each $x \in G$, let $\mathrm{RW}(x, \mu)$ be the law of a random walk on G started at x and with **step-length distribution** μ.

That is, $\mathrm{RW}(x, \mu)$ is the law of a process $(S_n^x)_{n=0}^{\infty}$ with

$$S_n^x = x + X_1 + X_2 + \cdots + X_n, \qquad 0 \leqslant n < \infty,$$

where $(X_i)_{i=1}^{\infty}$ are i.i.d. random elements in G with distribution μ.

Exact Coupling - Setup

Definition (M [8], 2017.)

Let the **successful exact coupling set** G_s be the set of $x \in G$ such that there exists a successful exact coupling of $\mathrm{RW}(0, \mu)$ and $\mathrm{RW}(x, \mu)$.

Exact Coupling - Results

Theorem (M [8], 2017.)

The successful exact coupling set is given by

$$G_s = \{x \in G : \exists n \geqslant 1, \mu^n \wedge \mu^n(x + \cdot) \neq \mathbf{0}\}.$$

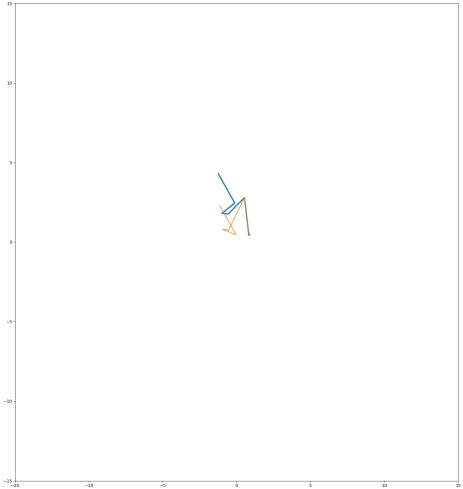
Moreover, for $x \in G_s$, $S \sim \mathrm{RW}(0, \mu), S^x \sim \mathrm{RW}(x, \mu)$, one has

$$\|\mathbf{P}(S_n \in \cdot) - \mathbf{P}(S_n^x \in \cdot)\|_{\mathrm{TV}} \in \begin{cases} O(1/\sqrt{n}) & \mathrm{ord}(x) = \infty \\ O(\rho^n), \rho \in (0,1) & \mathrm{ord}(x) < \infty. \end{cases}$$

The following notation is used for any measures ν, ν_1, ν_2:

- $(\nu_1 \wedge \nu_2)(B) := \sup\{\lambda(B) : \lambda \text{ is a measure, } \lambda \leqslant \nu_1, \nu_2\}$.
- $\nu^n :=$ n-fold convolution of ν with itself, i.e., the law of an i.i.d. sum of n ν-distributed RVs.
- $\nu(x + \cdot) :=$ shift of ν by $-x$.

Exact Coupling - Demo



Doeblin Trees

In this portion, one deals with a discrete time Markov chain.

Imagine starting a copy of the Markov chain from every possible state at every possible time such that when two paths meet, they merge.

Doeblin Trees - Definition

Definition (BHM [9], 2018.)

A **Doeblin graph** is a random graph determined by a **source of randomness** $(\xi_t)_{t\in\mathbb{Z}}$ and a **pathwise transition generator** h.

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- $(\xi_t)_{t\in\mathbb{Z}}$ is i.i.d. defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and takes values in some measurable space Ξ, and
- $h : S \times \Xi \to S$, where S is a countable set.

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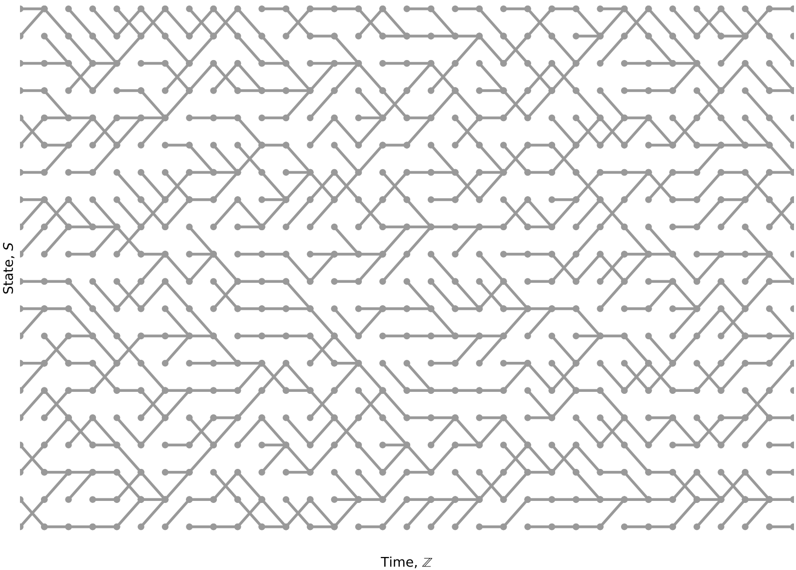
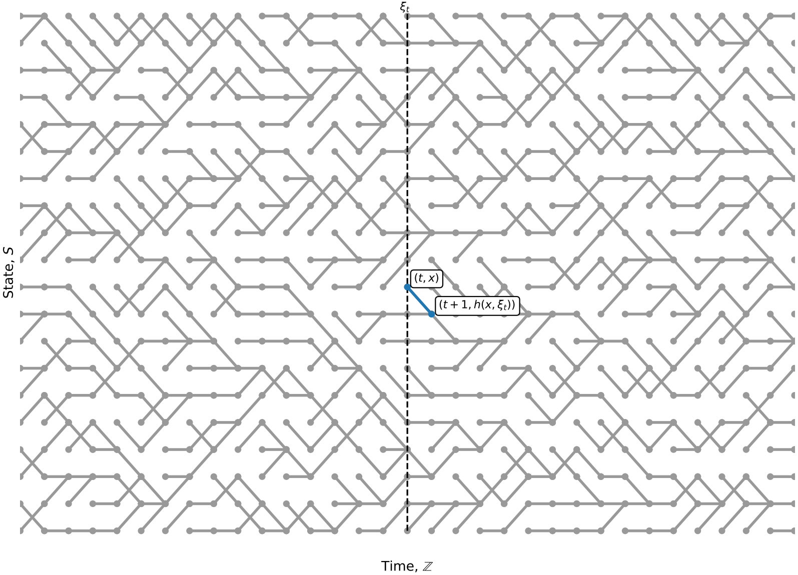
The Doeblin graph $\mathbf{G}$ determined by $(\xi_t)_{t\in\mathbb{Z}}$ and h is defined to have vertices $V(\mathbf{G}) := \mathbb{Z} \times S$ and edges determined by

$$(t, x) \mapsto (t+1, h(x, \xi_t)), \qquad t \in \mathbb{Z}, x \in S.$$

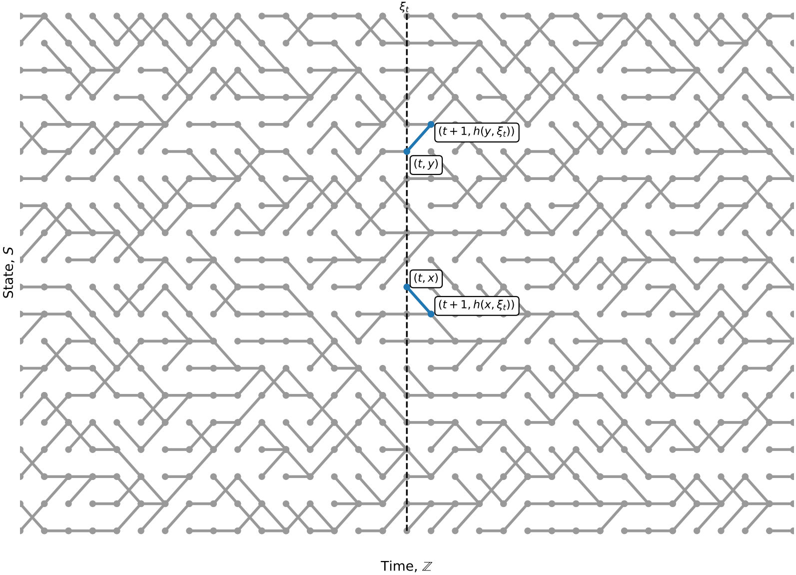
Doeblin Trees - Example



Doeblin Trees - Example



Doeblin Trees - Example

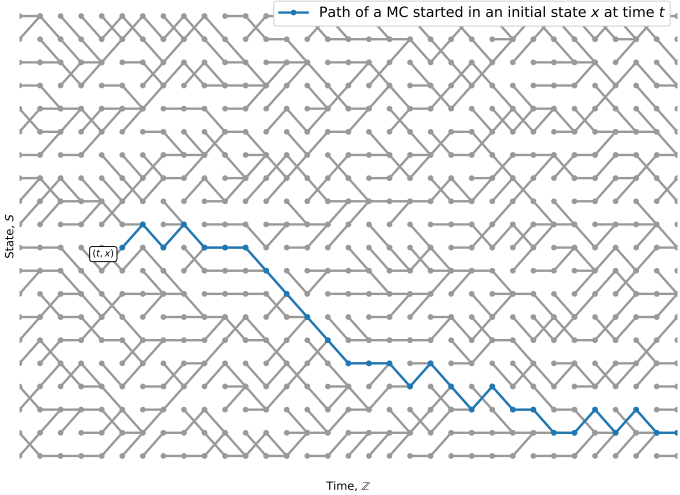


Doeblin Trees

A Doeblin graph $\mathbf{G}$ induces a transition matrix P defined by

$$P(x,y) := \mathbf{P}(h(x,\xi_0) = y), \qquad x, y \in S.$$

One may choose $h, (\xi_t)_{t\in\mathbb{Z}}$ so P is any desired transition matrix.



Doeblin Trees - History

This thesis defines a “Doeblin graph”, but such graphs have been implicitly studied in the past.

- Borovkov & Foss, 1992: Introduced the mathematical framework of stochastically recursive sequences.
- Propp & Wilson, 1996: Introduced a perfect sampling algorithm called Coupling From The Past (CFTP) to obtain a perfect sample from the stationary distribution of MC.
- Foss & Tweedie, 1998: Determined when CFTP succeeds using vertical coupling times.
- 20+ papers 1996–present: study, improve, or apply CFTP algorithm to specific MC.

Doeblin Trees - Key Ideas

This thesis studies the Doeblin graph itself, instead of the CFTP algorithm. It focuses on:

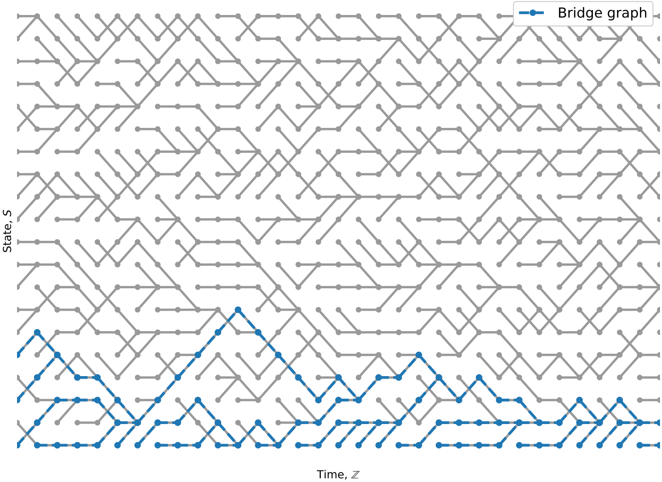
- Infinite state space case, where one doesn't already know how to do perfect sampling,
- Bridge sub-graph and unimodularity,
- Bi-recurrence.

For the rest of the talk, **G** is a Doeblin graph determined by $(\xi_t)_{t\in\mathbb{Z}}$ and $h : S \times \Xi \to S$, and the induced transition matrix P is assumed to be irreducible, aperiodic, and positive recurrent.

Doeblin Trees - Bridge Graph

Definition (BHM [9], 2018.)

Fix $x \in S$. The **bridge graph $\mathbf{B}(x)$ for state** x is the subgraph of $\mathbf{G}$ induced by all paths started in state x.



Doeblin Trees - Bridge Graph

For the remainder of the talk, $\mathbf{B}$ denotes a bridge graph $\mathbf{B}(x^*)$ for some fixed $x^* \in S$.

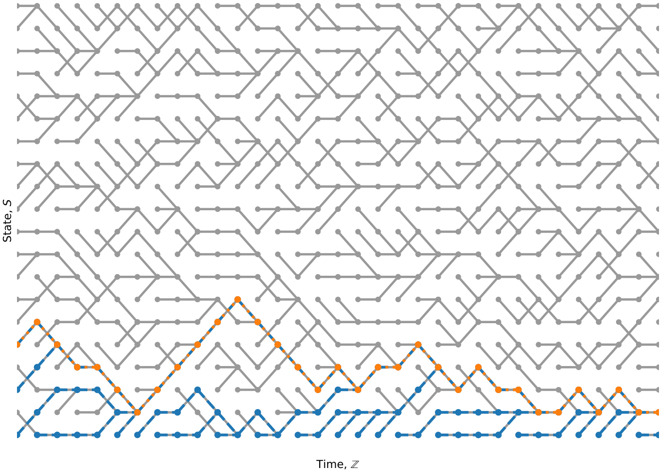
Theorem (BHM [9], 2018.)

Suppose $\mathbf{B}$ *is a tree. Then* $\mathbf{B}$ *is unimodularizable. That is, there exists a unimodular random rooted network* $[\mathbf{\Gamma}, \boldsymbol{o}]$ *such that*

$$[\mathbf{\Gamma}] \stackrel{d}{=} [\mathbf{B}].$$

Doeblin Trees - Structure of Bridge Graph

Since **B** is unimodularizable, classification theorem tells us that **B** must have a unique bi-infinite path in it (I/F type). **B** can then be thought of as a bi-infinite path $(t, \beta_t)_{t \in \mathbb{Z}}$ with a finite tree Q_t hanging from (t, β_t) for each t.



Doeblin Trees - Bi-recurrence

Definition (BHM [9], 2018.)

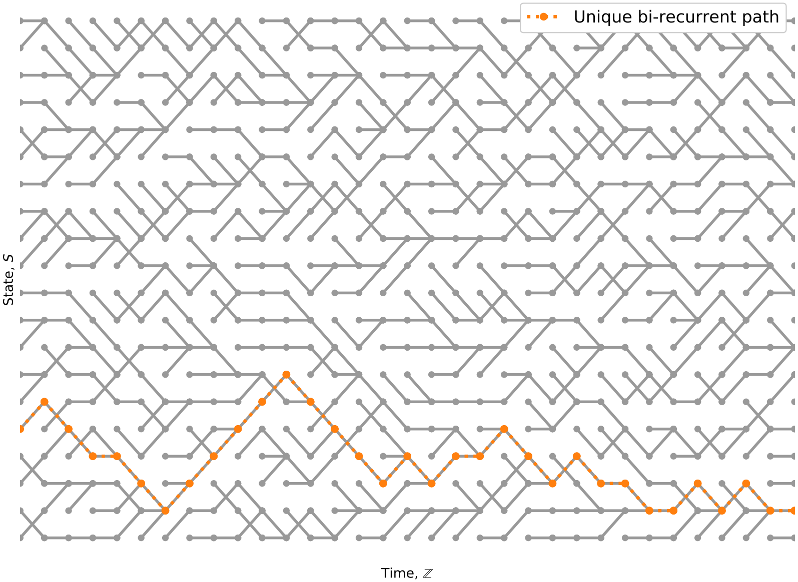
A sequence $(x_t)_{t\in\mathbb{Z}}$ in S is called **bi-recurrent** if $\{t \in \mathbb{Z} : x_t = x\}$ is unbounded above and below for all $x \in S$.

Theorem (BHM [9], 2018.)

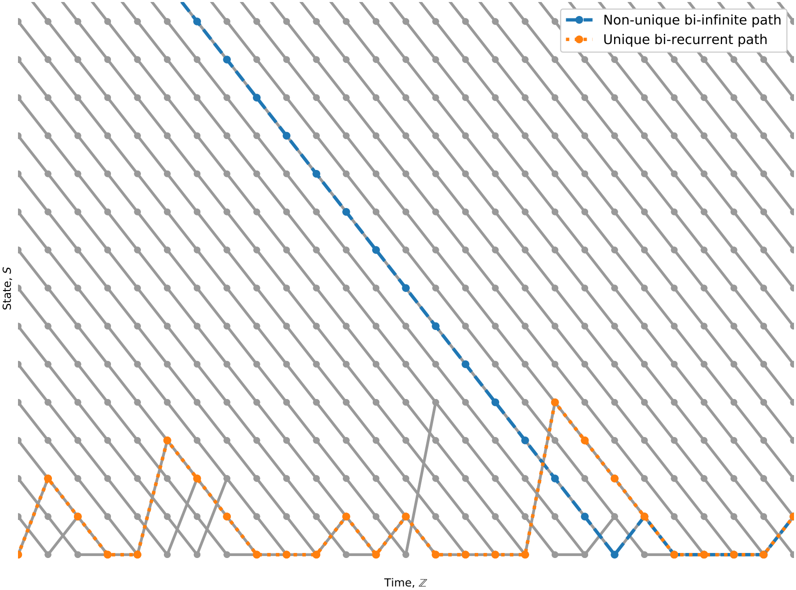
Suppose **G** *is a tree. Then a.s. there exists a unique path $(t, \beta_t)_{t\in\mathbb{Z}}$ in* **G** *such that $(\beta_t)_{t\in\mathbb{Z}}$ is bi-recurrent. Moreover, $(\beta_t)_{t\in\mathbb{Z}}$ is a stationary MC with transition matrix P.*

Here is where using a countably infinite state space is fruitful. If the state space is finite, the bi-recurrent path is the only bi-infinite path.

Doeblin Trees - Example



Doeblin Trees - Another Example



Doeblin Trees - Application to MC

By embedding Markov chains inside Doeblin trees, one obtains the following.

Theorem (BHM [9], 2018.)

Suppose that $(X_t)_{t\in\mathbb{Z}}$ *is a MC on a countable state space with irreducible, aperiodic, and positive recurrent transition matrix. Then* $(X_t)_{t\in\mathbb{Z}}$ *is stationary if and only if it is a.s. bi-recurrent.*

Doeblin Trees - Conservation Laws

Recall $\mathbf{B} = \mathbf{B}(x^*)$ for a fixed state $x^* \in S$.

Proposition (BHM [9], 2018.)

Each of the following has the same mean:

- $\#\{(t, x) \in V(\mathbf{B}) : t = 0\}$.
- $\#\{(t, x) \in V(\mathbf{B}) : (t, x)$ *first returns to* x^* *at time* $0\}$.
- $\#Q_0$ *(assuming* $\mathbf{B}$ *is a tree).*

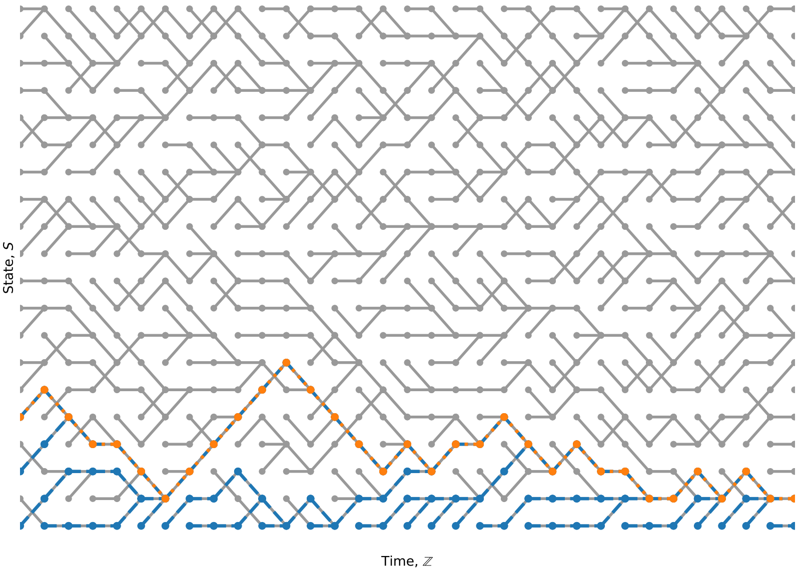
Moreover, this mean is at most $1/\pi(x^*)$*, where* π *is the stationary distribution corresponding to* P*.*

In the above, each bullet represents a different partition of $\mathbf{B}$:

- Vertices partitioned by their relative time (vertical slice).
- Vertices partitioned by their return times to x^*.
- Vertices partitioned by their merge times with bi-recurrent path.

Result obtained by mass-transports. E.g. send mass from each vertex in a vertical slice to the first time it returns to state x^*.

Doeblin Trees - Final Demo



Thank you

Questions?

References - Exact Coupling

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[10] Murphy, J.: Point-shifts of Point Processes on Topological Groups (2018).
Submitted. arXiv preprint.

Future Work

(i) Exact Coupling of Random Walks
- Are TV bounds within a multiplicative constant of optimal? What is the optimal exact coupling?
- Can results be extended to continuous time?
- What is the Hausdorff dimension of G_s?

(ii) Doeblin Trees
- Is there an algorithm to compute β_0 when S is infinite?
- Unimodularity came from the stationarity of $(\xi_t)_{t \in \mathbb{Z}}$, can the construction be generalized to other structures joined in a stationary manner?
- What changes in the null-recurrent and transient cases?

(iii) Point-shifts of Point Processes on Groups
- Can all stationary point processes on unimodular groups be seen as embeddings of unimodular networks?
- When can a unimodular network be embedded as a stationary point process?