Exact Coupling of Random Walks

on Polish Groups

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Introduction

Main question: How does changing the starting location of a (discrete time) random walk change its long-term behavior?

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- Sometimes it doesn't: $N(0,1)$ steps on $\mathbb{R}$
- Sometimes it does: symmetric steps of size 2 on $\mathbb{Z}$, always stays on evens/odds

Introduction

Let G be an Abelian Polish group. Let $S = \{S_n\}_{n=0}^{\infty}$ be a random walk on G started at 0 with **step-length distribution** μ,

$$S_n = X_1 + X_2 + \cdots + X_n, \qquad 0 \leq n < \infty$$

where $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random elements in G with distribution μ.

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where $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random elements in G with distribution μ. Let $S^x = \{S_n^x\}_{n=0}^{\infty}$ be a version of S started at $x \in G$,

$$S_n^x = x + X_1' + X_2' + \cdots + X_n', \qquad 0 \le n < \infty$$

where $\{X_i'\}_{i=1}^{\infty}$ are also i.i.d with distribution μ.

Introduction

Main question rephrased: For what values of $x \in G$ does

$$\|\mathbb{P}(S_n \in \cdot) - \mathbb{P}(S_n^x \in \cdot)\|_{TV} \to 0$$

as $n \to \infty$? Here $\|\nu\|_{TV} := \sup_F \nu(F) - \inf_F \nu(F)$ is total variation norm.

Introduction

Equivalent to admitting a **successful exact coupling**: define S and S^x on a common space $(\Omega, \mathcal{F}, \mathbb{P})$ such that there is an a.s. finite time T with

$$S_n = S_n^x, \qquad n \geq T.$$

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The coupling inequality shows obvious direction,

$$|\mathbb{P}(S_n \in \cdot) - \mathbb{P}(S_n^x \in \cdot)| \leq \mathbb{P}(T > n)$$

gives uniform bound in n. Other direction is tricky, but known in general.

What was known

Theorem (Herrmann, 1965, and Stam, 1966)

$G = \mathbb{R}$. S and S^x admit a successful exact coupling for all $x \in \mathbb{R}$ iff the step-lengths are spread out, i.e. if there is $n \geq 1$ such that $\mathbb{P}(X_1 + \cdots + X_n \in \cdot) \geq \int_\cdot f \, d\lambda$ for some Borel $f \geq 0$ not Lebesgue-a.e. zero.

Theorem (Stam, 1966)

$G = \mathbb{R}$. Suppose μ is purely atomic with A the set of atoms of μ. Then S and S^x admit a successful exact coupling iff x is in the additive subgroup generated by $A - A$.

In 2011 Thorisson asked for people to fill in the middle ground.

Results

Theorem (Murphy, 2017 (submitted))

S and S^x admit a successful exact coupling iff for some $n \geq 1$

$$\mu^n \wedge \mu^n(x + \cdot) \neq 0.$$

Here $\nu_1 \wedge \nu_2$ is the largest measure smaller than ν_1, ν_2, and μ^n is the n-fold convolution of μ, i.e. the law of $X_1 + \cdots + X_n$.

Results

Theorem (Murphy, 2017 (submitted))

If G is locally compact with Haar measure λ, and if S and S^x admit a successful exact coupling for all $x \in G$, then μ is **spread out***, i.e. $\mu^n \geq \int_{\cdot} f \, d\lambda$ for some Borel $f \geq 0$ not λ-a.e. zero. If G is connected, the converse holds as well.*

Results

Theorem (Murphy, 2017 (submitted))

If μ is purely atomic with A the set of atoms of μ, then S and S^x admit a successful exact coupling iff x is in the subgroup generated by $A - A$.

Results

Theorem (Murphy, 2017 (submitted))

The set of x *for which* S *and* S^x *admit a successful coupling is a Borel measurable subgroup of* G*.*

The paper

Many of these results hold with extra conditions on non-Abelian groups as well. See paper on arXiv for details.

References

- [1] Arnaldsson, Ö.: On coupling of discrete random walks on the line (2010)
- [2] Berbee, H.C.: Random walks with stationary increments and renewal theory.
  MC Tracts **112**, 1–223 (1979)
- [3] Herrmann, H.: Glättungseigenschaften der Faltung (1965)
- [4] Stam, A.: On shifting iterated convolutions i.
  Compositio Math **17**, 268–280 (1966)
- [6] Thorisson, H.: Coupling, stationarity, and regeneration, vol. 200.
  Springer New York (2000)
- [7] Thorisson, H.: Open problems in renewal, coupling and palm theory.
  Queueing Systems **68**(3-4), 313–319 (2011)