Exact Coupling of Random Walks on Polish Groups (arXiv:1706.06968)

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Main Question

How does changing the starting location of a (discrete time) random walk change its long-term behavior?

Starting location doesn't matter for:

- $N(0,1)$ steps on $\mathbb{R}$,
- if the random walk has stationary distribution.

Starting location does matter for:

- symmetric steps of size 2 on $\mathbb{Z}$, because the walk will always stay on evens or always stay on odds.

Introduction

Let G be an Abelian Polish group. Let $S = \{S_n\}_{n=0}^{\infty}$ be a random walk on G started at 0 with **step-length distribution** μ,

$$S_n = X_1 + X_2 + \cdots + X_n, \qquad 0 \leqslant n < \infty$$

where $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random elements in G with distribution μ.

Let $S^x = \{S_n^x\}_{n=0}^{\infty}$ be a version of S started at $x \in G$,

$$S_n^x = x + X_1' + X_2' + \cdots + X_n', \qquad 0 \leqslant n < \infty$$

where $\{X_i'\}_{i=1}^{\infty}$ are also i.i.d with distribution μ.

Then one is interested in for what values of $x \in G$ does

$$\|\mathbf{P}(S_n \in \cdot) - \mathbf{P}(S_n^x \in \cdot)\|_{TV} \to 0$$

as $n \to \infty$? Here $\|\nu\|_{TV} := \sup_F \nu(F) - \inf_F \nu(F)$ is total variation.

Total variation convergence in this setting is equivalent to admitting a **successful exact coupling**: define S and S^x on a common space $(\Omega, \mathcal{F}, \mathbf{P})$ such that there is an a.s. finite time T with

$$S_n = S_n^x, \qquad n \geqslant T.$$

The coupling inequality shows obvious direction,

$$|\mathbf{P}(S_n \in \cdot) - \mathbf{P}(S_n^x \in \cdot)| \leqslant \mathbf{P}(T > n),$$

which gives a uniform bound in terms of n. The other direction is tricky, but known in general.

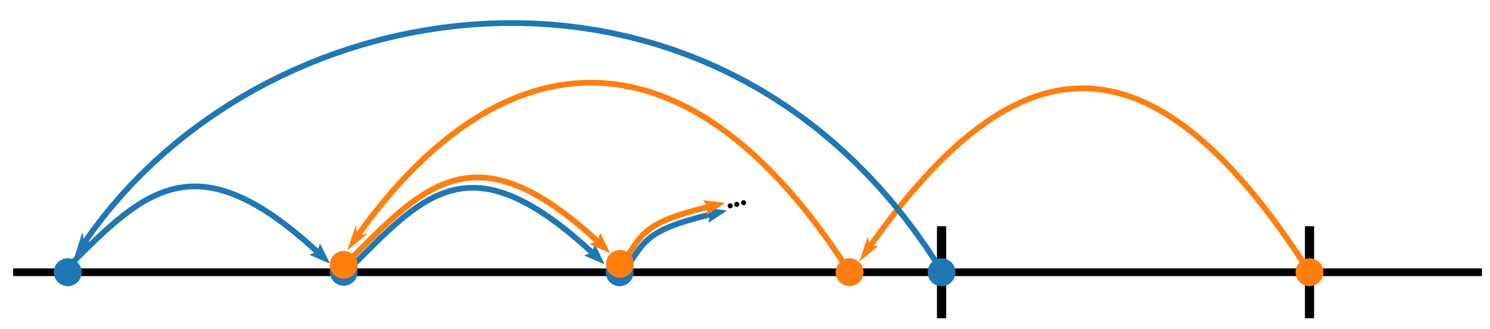

Figure 1: A realization of a successful exact coupling. Two random walks started in different locations (marked by vertical bars) eventually merge.

Main Question Reframed

For which $x \in G$ does there exist a successful exact coupling of S and S^x?

Theorem (Results)

(a) *S and S^x admit a successful exact coupling iff there is $n \geqslant 1$ such that $\mu^n \wedge \mu^n(x + \cdot) \neq 0$,*

(b) *if G is locally compact with Haar measure λ, and if S and S^x admit a successful exact coupling for all $x \in G$, then μ is spread out, i.e. there is $n \geqslant 1$ such that $\mu^n \geqslant \int_\cdot f \, d\lambda$ for some Borel $f \geqslant 0$ not λ-a.e. zero. If G is connected, the converse holds as well.*

(c) *if μ is purely atomic with A the set of atoms of μ, then S and S^x admit a successful exact coupling iff x is in the subgroup generated by $A - A$.*

(d) *the set of x for which S and S^x admit a successful exact coupling is a Borel measurable subgroup of G.*

Note: (b) and (c) were already known in the case $G = \mathbb{R}$. Results for non-Abelian G appear in the arXiv paper.

Proof Idea

If $\mu^n \geqslant \nu + \nu(x + \cdot)$ for some nonzero measure ν, one may construct a coupling of (S, S^x) for which the difference walk $\{S_{kn} - S_{kn}^x\}_{k \in \mathbb{N}}$ is a symmetric random walk on the cyclic subgroup generated by x. Thus the analysis reduces to that of $\mathbb{Z}$ or $\mathbb{Z}/d\mathbb{Z}$. It then suffices to characterize when such a measure ν exists.

References

[1] Hermann Thorisson.
Open problems in renewal, coupling and palm theory.
Queueing Systems, 68(3-4):313–319, 2011.

[2] Örn Arnaldsson.
On coupling of discrete random walks on the line, 2010.

[3] Hermann Thorisson.
Coupling, stationarity, and regeneration, volume 200.
Springer New York, 2000.

[4] Henry CP Berbee.
Random walks with stationary increments and renewal theory.
MC Tracts, 112:1–223, 1979.

[5] AJ Stam.
On shifting iterated convolutions i.
Compositio Math, 17:268–280, 1966.

[6] Horst Herrmann.
Glättungseigenschaften der Faltung.
1965.

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