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M427J Quiz 6 Solutions

Problem 1. [3 pts] Write the eigenvalues of the matrix T . Hint: no work is required.

$$T = \begin{pmatrix} 1 & 3 & 1 \\ & 2 & 2 \\ & & 1 \end{pmatrix}$$

Solution. For triangular matrices, the eigenvalues are on the diagonal, $\lambda = 1, 2, 1$.

Problem 2. [5 pts] Suppose A is a 3×3 real matrix whose only eigenvalue is 0. Also assume that $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for the set of solutions to $A\vec{x} = 0$, and that $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for the set of solutions to $A^2\vec{x} = 0$. You are given that $A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$. Find the general solution to $\frac{d}{dt}\vec{x} = A\vec{x}$.

Solution. We get two independent solutions from the eigenvectors for $\lambda = 0$,

$$\vec{x}_1(t) = e^{0t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_2(t) = e^{0t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We get the last independent solution from the generalized eigenvector of rank 2 for $\lambda = 0$,

$$\vec{x}_3 = e^{At} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = I \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + tA \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 + 0 + \dots = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 4t \\ 1 + 2t \\ 2t \end{pmatrix}.$$

Thus the general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 + 4t \\ 1 + 2t \\ 2t \end{pmatrix}$$

Problem 3. [2 pts]

(a) In one line, show that if \vec{x} is a solution to $\frac{d}{dt}\vec{x} = A\vec{x}$, then so is $\vec{y} = \frac{d}{dt}\vec{x}$.

Solution.

$$\frac{d}{dt}\vec{y} = \frac{d}{dt} \left(\frac{d}{dt}\vec{x} \right) = \frac{d}{dt}(A\vec{x}) = A \frac{d}{dt}\vec{x} = A\vec{y}$$

(b) Using part (a), prove that if $\vec{x} = e^{\lambda t}\vec{B}(t)$ is a solution to $\frac{d}{dt}\vec{x} = A\vec{x}$, then so is $\vec{y} = e^{\lambda t}\frac{d}{dt}\vec{B}(t)$. Hint: the product rule from calculus holds even though $\vec{B}(t)$ is a vector function. Hint 2: linear combinations of solutions are solutions.

Proof. If $\vec{x} = e^{\lambda t}\vec{B}(t)$ is a solution, then by part (a), so is

$$\frac{d}{dt}\vec{x} = \lambda e^{\lambda t}\vec{B}(t) + e^{\lambda t}\frac{d}{dt}\vec{B}(t).$$

But then

$$e^{\lambda t}\frac{d}{dt}\vec{B}(t) = \frac{d}{dt}\vec{x} - \lambda\vec{x}$$

is a linear combinations of solutions, and is thus itself a solution. □