NAME: ____________________

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M427J Quiz 5 Solutions

Problem 1. **[5 pts]** Give a basis of $\mathbb{R}^2$ that includes the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Explain why your answer is a basis.

Any answer of the form $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\}$ where $a \neq b$, e.g. $\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}}$. It is a basis because it is an independent set of size 2 inside $\mathbb{R}^2$, which has dimension 2.

Problem 2. **[5 pts]** Let V be the set of continuous functions f such that $\int_0^1 f(x)\,dx = 0$. Using the subspace lemma, show that V is a vector space.

Proof.

1. The zero function is continuous and $\int_0^1 0\,dx = 0$, so $0 \in V$.
2. If $f, g \in V$ then f, g are continuous, so $f + g$ is continuous and
$$\int_0^1 (f(x) + g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = 0 + 0 = 0.$$
Thus $f + g \in V$.
3. If $f \in V, \alpha \in \mathbb{R}$, then f is continuous, so αf is continuous and
$$\int_0^1 \alpha f(x)\,dx = \alpha \int_0^1 f(x)\,dx = \alpha \cdot 0 = 0.$$
Thus $\alpha f \in V$.

It follows by the subspace lemma that V is a subspace of $\mathfrak{F}$. □

Note: You could have saved some writing by stating that the C^0, the space of continuous functions, is a vector space, and using the subspace lemma with C^0 as the big space instead of $\mathfrak{F}$.

Problem (Bonus). **[2 pts]** Do the previous problem again by giving a linear operator L such that $V = \ker L$. Make sure to specify the domain and codomain of L.

Proof. Let $L : C^0 \to \mathbb{R}$ be defined by $L(f) = \int_0^1 f(x)\,dx$ for each $f \in C^0$. Then L is linear and $V = \ker L$, hence V is a subspace of C^0. □