NAME: ____________________

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# M427J Quiz 7 Solutions

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**Problem 1.** [**4 pts**] In this question $A$ is a real matrix. The parts should be considered separately. Use the given information to determine whether solutions to $\dot{x} = Ax$ are asymptotically stable, stable (but not asymptotically stable), or unstable. No work is required for this problem.

(a) $A$ is $4 \times 4$ and has eigenvalues $-\sqrt{2}, -5, -\pi + 2i, -\pi - 2i$.
Choose one: **[asymptotically stable]** / stable / unstable
Real parts of eigenvalues are all strictly negative.

(b) $A$ has characteristic polynomial $-\lambda^3$ and there are 3 linearly independent solutions to $Ax = 0$.
Choose one: asymptotically stable / **[stable]** / unstable
Not asymptotically stable since 0 is an eigenvalue, but a full set of three eigenvectors means solution is stable since the real part of 0 isn't strictly positive.

(c) $A$ has characteristic polynomial $-(\lambda - 4)(\lambda^2 + 1)$.
Choose one: asymptotically stable / stable / **[unstable]**
$\lambda = 4$ eigenvalue has strictly positive real part.

(d) $A$ is $3 \times 3$ and has only one eigenvalue $-3$. The equation $Av = -3v$ has only one linearly independent solution.
Choose one: **[asymptotically stable]** / stable / unstable
Real parts of eigenvalues are all strictly negative.

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**Problem 2.** [**6 pts**] For $\alpha \neq 0$ the heat equation in two spacial dimensions is

$$u_t = \alpha^2(u_{xx} + u_{yy})$$

Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, find ordinary differential equations satisfied by $X, Y$, and $T$. You may assume $X, Y$, and $T$ are never zero. Hint: $F(t) = G(x, y)$ implies both sides are constant.

**Solution.** Since $u_t = XYT', u_{xx} = X''YT, u_{yy} = XY''T$ plugging in to the equation gives

$$XYT' = \alpha^2(X''YT + XY''T) \implies \frac{T'}{\alpha^2 T} = \frac{X''}{X} + \frac{Y''}{Y}$$

$$\implies \boxed{\frac{T'}{\alpha^2 T} = \lambda}, \quad \frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

$$\implies \frac{X''}{X} = \lambda - \frac{Y''}{Y}$$

$$\implies \boxed{\frac{X''}{X} = \mu}, \quad \boxed{\lambda - \frac{Y''}{Y} = \mu}.$$

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**Problem (Bonus)** [**2 pts**] The point of this question is to show that differentiable functions are not made up of too many high frequencies. Suppose $f$ and $f'$ are continuous on $[-1, 1]$. In this case the Fourier coefficients for $f$ are given by

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx, \quad b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx.$$

On the back of your quiz, show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$. Hint: integrate by parts (differentiate $f$), then use that sine and cosine are bounded.[1]

[1] In fact, a much stronger statement is true: if the first $k \geq 0$ derivatives of $f$ are continuous on $[-l, l]$, then the Fourier coefficients for $f$ on $[-l, l]$ satisfy $\lim_{n\to\infty} n^k a_n = \lim_{n\to\infty} n^k b_n = 0$. That is, the smoother $f$ is, the faster it's Fourier coefficients decay. This is much harder to prove.

**Proof.** Integrating by parts

$$a_n = \left[ f(x) \frac{\sin(n\pi x)}{n\pi} \right]_{-1}^{1} - \int_{-1}^{1} f'(x) \frac{\sin(n\pi x)}{n\pi} \, dx.$$

Applying the triangle inequality, bringing absolute values inside the integral, and using that sine is bounded by 1,

$$\begin{aligned} |a_n| &\leq \frac{1}{n\pi} \left( |f(1)| + |f(-1)| + \int_{-1}^{1} |f'(x)| \, dx \right) \\ &\leq \frac{1}{n\pi} (\max |f| + \max |f| + 2 \max |f'|) \\ &\to 0. \end{aligned}$$

The proof for $b_n$ is nearly identical. □