Point-shifts of Point Processes on Topological Groups (arXiv:1704.08333)

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Introduction

Stationary point processes $\mathfrak{m}$ are models of discrete subsets of points that exhibit "statistical homogeneity". That is, their distributions are invariant with respect to translations, or, more generally, with respect to the action of a group.
Consider a covariant point-shift $\mathfrak{H}$ on a point process $\mathfrak{m}$ on a group $\mathbb{X}$. $\mathfrak{H}$ maps points of $\mathfrak{m}$ to other points of $\mathfrak{m}$ in such a way that $\mathfrak{H}$ commutes with shifting the point-process.

Definition (Graph generated by a point-shift)

The graph $G^{\mathfrak{H}}$ is generated by considering each point of $\mathfrak{m}$ a vertex and having directed edges from each point to its image under $\mathfrak{H}$.

Consider the point process to be "rooted" at the identity under its Palm probability $\mathbf{P}^{\mathfrak{m}}$. Then $\mathfrak{m}$ behaves in many respects like a unimodular network, and $G^{\mathfrak{H}}$ like the graph generated by a vertex-shift on a unimodular network. When the underlying group is $\mathbb{R}^d$, many such connections are known, such as analogs of Mecke's invariance theorem, the no finite flow-adapted selection theorem, the cardinality classification of connected components of $G^{\mathfrak{H}}$, and more. This paper investigates the ways that these connections generalize to other groups. The main tool is the mass transport principle for point processes.

In this poster, we focus on the graph $G^{\mathfrak{H}}$ and how the cardinality classification of connected components of it generalizes (and when it does not generalize). The arXiv paper investigates the connection with unimodular networks much more closely, and hones in on the boundary of when certain results generalize.

Partitions of $G^{\mathfrak{H}}$

In studying the graph $G^{\mathfrak{H}}$, there are two partitions of points that are considered.

- The **component partition** $\mathcal{C}^{\mathfrak{H}}$, where X and Y are in the same partition element if they are in the same undirected graph component of $G^{\mathfrak{H}}$. Equivalently, X and Y are in the same component C if $\mathfrak{H}^n(X) = \mathfrak{H}^m(Y)$ for some n, m.
- The **foil partition** $\mathcal{L}^{\mathfrak{H}}$, where X and Y are in the same partition element if they their trajectories under $\mathfrak{H}$ eventually merge. Equivalently, X and Y are in the same foil L if $\mathfrak{H}^n(X) = \mathfrak{H}^n(Y)$ for some n. Thus the foil partition is a finer partition than the component partition.

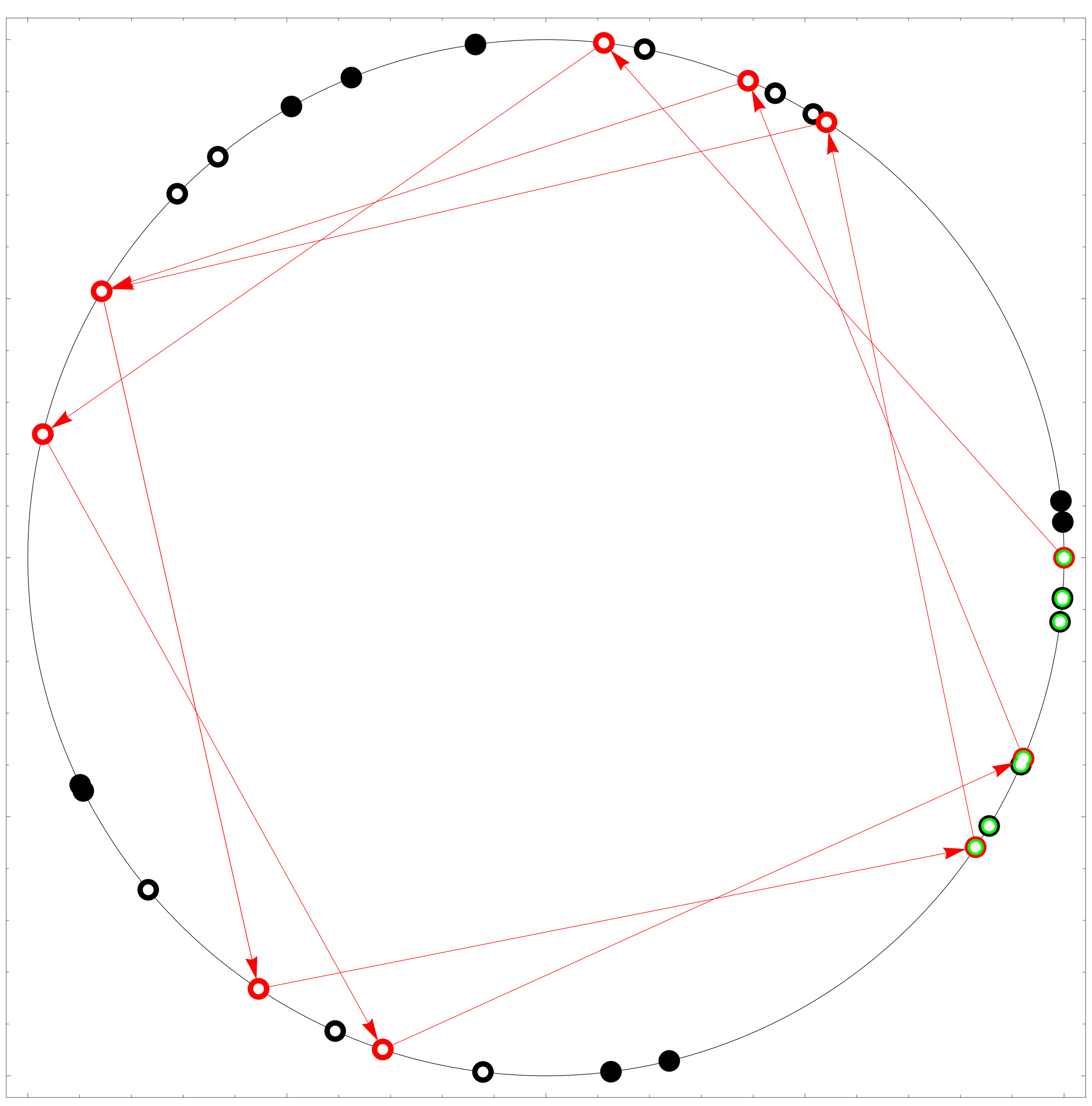

Figure 1: The trajectory of the identity of the circle group $\mathbb{X} = \{z \in \mathbb{C} : |z| = 1\}$ under a point-shift that sends each point X to the point Y closest to a 90° rotation of X. Points in the same component of $G^{\mathfrak{H}}$ as the identity are marked with a white dot. Points in the same foil of $G^{\mathfrak{H}}$ as the identity are further marked with green.

There is a stark contrast in behavior depending on whether the underlying group is unimodular. The cardinality classification, previously known for $\mathbb{X} = \mathbb{R}^d$, generalizes verbatim to unimodular $\mathbb{X}$.

Theorem (Cardinality Classification of a Component)

Suppose $\mathbb{X}$ is unimodular. Then $\mathbf{P}$-a.s. $G^{\mathfrak{H}}$ is locally finite, and each connected component C of $G^{\mathfrak{H}}$ is in one of the three following classes:

1. ***Class $\mathcal{F}/\mathcal{F}$:*** *C is finite, and hence so is each of its $\mathfrak{H}$-foils. In this case, when denoting by $1 \leqslant n = n(C) < \infty$ the number of its foils:*
   - *C has a unique cycle of length n;*
   - *$\mathfrak{H}^\infty(\mathfrak{m}) \cap C$ is the set of vertices of this cycle.*
2. ***Class $\mathcal{I}/\mathcal{F}$:*** *C is infinite and all of its $\mathfrak{H}$-foils are finite. In this case:*
   - *C is acyclic;*
   - *Each foil has a junior foil;*
   - *$\mathfrak{H}^\infty(\mathfrak{m}) \cap C$ is a unique **bi-infinite** path, i.e. a sequence $\{X_n\}_{n\in\mathbb{Z}}$ of points of $\mathfrak{m}$ such that $\mathfrak{H}(X_n) = X_{n+1}$ for all n.*
3. ***Class $\mathcal{I}/\mathcal{I}$:*** *C is infinite and all its $\mathfrak{H}$-foils are infinite. In this case:*
   - *C is acyclic;*
   - *$\mathfrak{H}^\infty(\mathfrak{m}) \cap C = \emptyset$.*

On the other hand, the classification fails in many respects for non-unimodular groups. Consider the (non-unimodular) $ax + b$ group,

$$\mathbb{X} := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$$

with matrix multiplication and the topology inherited from $\mathbb{R}^4$. $\mathbb{X}$ is identified with the right half-plane in $\mathbb{R}^2$ by identifying (a, b) with $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Let $\mathfrak{m}$ be a Poisson point process of intensity 1 on $\mathbb{X}$.
For all $(a, b) \in \mathbb{X}$ define the strip

$$S(a, b) := [a, \infty) \times [b - \delta a, b + \delta a]$$

for some fixed $\delta \in (0, \frac{1}{2})$. One has $\mathbf{P}$-a.s. $\mathfrak{m}(S(X)) < \infty$ for all $X \in \mathfrak{m}$. This leads to the **strip point-shift** $\mathfrak{H}$ where $\mathfrak{H}(X)$ is defined to be the right-most point of $\mathfrak{m}$ in $S(X)$ for each $X \in \mathfrak{m}$.

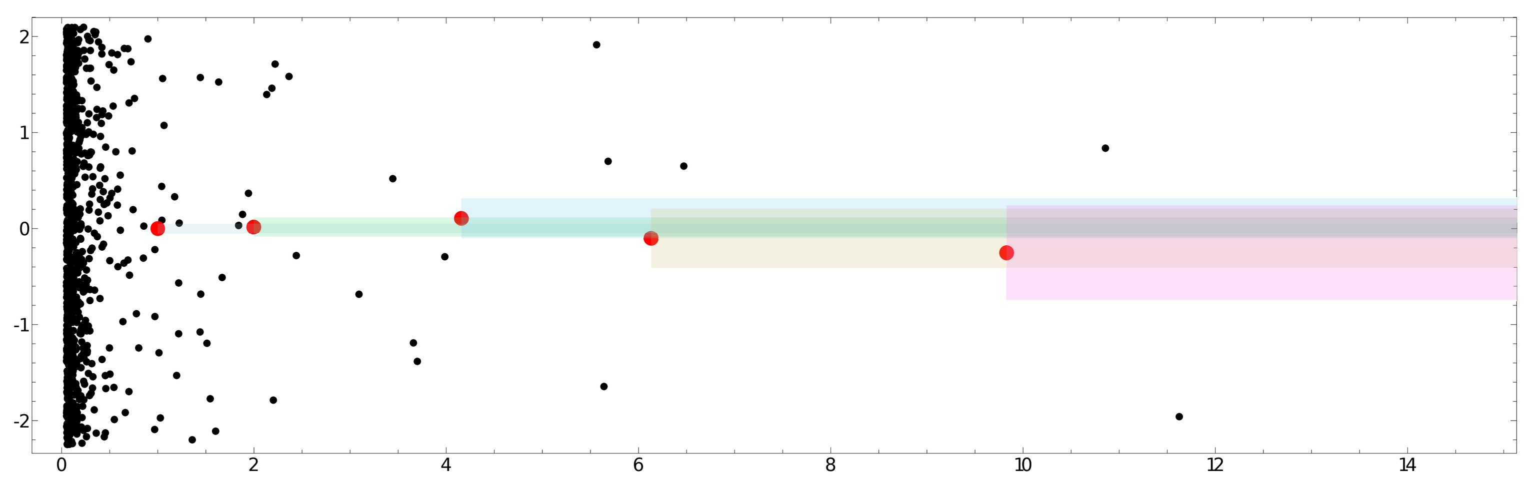

Figure 2: Iterates of the identity under the strip point-shift on the $ax + b$ group.

For the strip point-shift, it holds that the foils and connected components are identical because every component contains a fixed point, and the foils and components are in bijection with the fixed points of $\mathfrak{H}$. The connected component of a fixed point Y of $\mathfrak{H}$ is all $X \in \mathfrak{m}$ that are eventually sent to Y. All components and foils are infinite (class $\mathcal{I}/\mathcal{I}$). However, the components are not acyclic and $\mathfrak{H}^\infty(\mathfrak{m}) = \{X \in \mathfrak{m} : \mathfrak{H}(X) = X\} \neq \emptyset$, contrary to what the classification theorem would suggest for unimodular $\mathbb{X}$. It follows that the properties of the cardinality classification cannot be extended beyond the case of unimodular $\mathbb{X}$.

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