A BRIEF INTRODUCTION TO PROCESSES OF FLATS

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1. Introduction

This text was written to be given as a half-hour talk. It closely follows the proofs and examples in [1].

2. Processes of flats

Let $A(d,k)$ denote the space of k-dimensional planes in $\mathbb{R}^d$, equipped with the topology inherited by viewing $A(d,k)$ as a subset of $\mathcal{F}$, the space of closed sets equipped with the Fell topology. We call $A(d,k)$ the space of **k-flats**, and a point process on $A(d,k)$ a **process of k-flats** or a **k-flat process**.

Examples

1. Poisson point process. Since $\mathcal{F}$ is locally compact (in fact compact), second countable, and Hausdorff, for any Radon intensity measure Λ on $\mathcal{F}$, there is a Poisson point process on $\mathcal{F}$ with intensity Λ. Taking Λ to be concentrated on $A(d,k)$ gives the existence of many k-flat Poisson processes.

2. Sections with a fixed plane. From linear algebra we know that the intersection of a k-flat and a j-flat is either empty or an i-flat for some $i \leqslant \min\{k,j\}$, where a 0-flat is defined to be a point. Given a k-flat processes $\Phi = \sum_n \delta_{E_n}$ and a fixed j-flat S, we can ask when $\Phi \cap S := \sum_n \delta_{E_n \cap S} 1_{E_n \cap S \neq \emptyset}$ is itself an i-flat process for some i.

3. Intersection processes. Similarly, given a flat processes $\Phi = \sum_n \delta_{E_n}$ we can ask when certain intersections of flats of Φ form their own process of flats. Of particular interest are when Φ is a hyperplane process ($k = d-1$), one asks when $\Phi^{r\cap} := \sum_{n_1 \neq \cdots \neq n_r} \delta_{E_{n_1} \cap \cdots \cap E_{n_r}} 1_{E_{n_1} \cap \cdots \cap E_{n_r}}$ is a process of flats, or when Φ is a k-flat process with $k \geqslant d/2$ and one asks when $\Phi^{2\cap} := \sum_{n \neq m} \delta_{E_n \cap E_m} 1_{E_n \cap E_m \neq \emptyset}$ is a process of flats.

In the stationary case for point processes on $\mathbb{R}^d$ we have the Campbell-Little-Mecke-Matthes theorem, which allows us to decompose a Campbell-Matthes measure $\mathcal{C}$ into a product $\gamma\lambda \otimes \mathbb{P}^0$. For stationary k-flat processes, we have an analogous result that allows a decomposition of intensity measures.

Let $G(d,k)$ denote the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^d$, and let $\pi_0 : \cup_{k=1}^{d-1} A(d,k) \to \cup_{k=1}^{d-1} G(d,k)$ the map that sends a flat to its translate through the origin. We note that $G(d,k)$ is closed in $\mathcal{F}, \mathcal{F} \setminus \{\emptyset\}$, $A(d,k)$ is closed in $\mathcal{F} \setminus \{\emptyset\}$, and π_0 is continuous. We are now ready to state the fundamental decomposition result.

Theorem 2.1. *Let Θ be a translation invariant Radon measure concentrated on $A(d,k)$. Then there exists a uniquely determined finite measure Θ_0 concentrated on $G(d,k)$ such that*

$$\int_{A(d,k)} f \, d\Theta = \int_{G(d,k)} \int_{L^\perp} f(L+x) \, \lambda_{L^\perp}(dx) \, \Theta_0(dL) \tag{1}$$

for all measurable $f \geqslant 0$.

Proof. The proof is ultimately made possible by the fact that $G(d,k)$ is closed, and hence compact, in $\mathcal{F}$, allowing us to use local decompositions and add them. Fix $U \in G(d, d-k)$, and define

$$G_U := \{L \in G(d,k) : \dim(U \cap L) = 0\},$$
$$A_U := \{L + x, L \in G_U, x \in U\}.$$

The G_U and A_U are the pieces we will use to cover $G(d,k)$ and $A(d,k)$ respectively. Unsurprisingly, the map

$$\varphi : G_U \times U \to A_U$$
$$(L, x) \mapsto L + x$$

is a homeomorphism, a detail whose proof we omit. For fixed $A \in \mathcal{B}(G_U)$, $\Theta(\varphi(A \times \cdot))$ is a translation invariant Radon measure on U. To see that it is Radon, simply note that $\Theta(\varphi(A \times B)) \leqslant \Theta(\mathcal{F}_B)$. Thus

$$\Theta(\varphi(A \times \cdot)) = \rho(A)\lambda_U$$

for some constant $\rho(A)$. Now fixing $B \subseteq U$ compact with $\lambda_U(B) = 1$ and varying $A \in \mathcal{B}(G_U)$ we find $\rho(A) = \Theta(\varphi(A \times B))$ is a measure of total mass $\Theta(\varphi(G_U \times B)) \leqslant \Theta(\mathcal{F}_B) < \infty$. It follows that

$$\Theta \circ \varphi = \rho \otimes \lambda_U$$

and that for $f \geqslant 0$ a measurable function on $A(d,k)$

$$\int_{A_U} f \, d\Theta = \int_{G_U \times U} f \circ \varphi \, d(\Theta \circ \varphi) = \int_{G_U} \int_U f(L + x) \, \lambda_U(dx) \, \rho(dL).$$

For a fixed $L \in G_U$ let $\Pi_L : U \to L^\perp$ denote the orthogonal projection onto $L^\perp$. Note that $\ker(\Pi_L) = 0$ because $\dim(U \cap L) = 0$, and hence Π_L is bijective. Then

$$\int_U f(L + x)\lambda_U(dx) = \int_{L^\perp} f(L + \Pi_L^{-1}(y)) \, d\lambda_U(\Pi_L^{-1}(y)) = \int_{L^\perp} f(L + y) d\lambda_U(\Pi_L^{-1}(y)).$$

Here we used that $L + \Pi_L^{-1}(y) = L + y$ because shifting L in a direction already in L doesn't change anything. Next we recognize that since Π_L is linear, $\lambda_U \circ \Pi_L^{-1}$ is a translation invariant Radon measure on $L^\perp$, thus $\lambda_U \circ \Pi_L^{-1} = a(L)\lambda_{L^\perp}$ for some constant $a(L)$ depending continuously on L. Thus we have that

$$\int_{A_U} f \, d\Theta = \int_{G_U} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(dx) \, a(L)\rho(dL) = \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(dx) \, \Theta_U(dL)$$

where $\Theta_U(dL) := 1_{G_U}(L)a(L)\rho(dL)$. This completes the local portion of the proof.

Now we put the pieces together. The sets $\{G_U\}_{U \in G(d,d-k)}$ form an open cover in $G(d,k)$ of the compact $G(d,k)$. Indeed, If $L \in G(d,k)$, then $L \in G_{L^\perp}$. Moreover, if $\{L_n\}_n$ is a sequence in $G(d,k) \setminus G_U$ and $L_n \to L \in G(d,k)$, then there are unit vectors $u_n \in L_n \cap U$, and hence a convergent subsequence $u_{n_k} \to u$. We have $u \in U$ since U is closed, and $u \in L$ since $u_{n_k} \in L_{n_k}$ for each k. Since u is a unit vector, $\dim(L \cap U) > 0$ and $L \notin G_U$, showing $G(d,k) \setminus G_U$ is closed, and hence G_U is open in $G(d,k)$. Thus we may choose $U_1, \ldots, U_N \in G(d, d-k)$ such that $\cup_{i=1}^N G_{U_i} = G(d,k)$. Then $\cup_{i=1}^N A_{U_i} = A(d,k)$ and each A_{U_i} is translation invariant. We disjointify $A_1 := A_{U_1}, A_{i+1} := A_{U_{i+1}} \setminus \cup_{j=1}^i A_{U_j}$ for each i and note that the A_i are still translation invariant but now form a disjoint cover of $A(d,k)$. Apply the local portion of the proof to the translation invariant Radon measures $\Theta|_{A_i} := \Theta(A_i \cap \cdot)$ for each i to get measures $\Theta_1, \ldots, \Theta_N$ such that for $f \geqslant 0$ measurable

$$\int_{A_i} f d\Theta = \int_{A_{U_i}} f \, d(\Theta|_{A_i}) = \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(dx) \, \Theta_i(dL).$$

Then $\Theta_0 := \sum_{i=1}^N \Theta_i$ satisfies (1),

$$\int_{A(d,k)} f \, d\Theta = \int_{G(d,k)} \int_{L^\perp} f(L + x) \, \lambda_{L^\perp}(dx) \, \Theta_0(dL).$$

Taking $f := 1_{\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)}$ for any $A \in \mathcal{B}(A(d,k))$ then gives

$$\Theta(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A)) = \kappa_{d-k}\Theta_0(A) \tag{2}$$

where k_{d-k} is the volume of a $d-k$ dimensional ball. This show the uniqueness of Θ_0. ☐

Applying the theorem to the intensity measure of a point process, we immediately get the following.

Corollary 2.2. *Let Φ be a stationary k-flat processes in $\mathbb{R}^d$ with Radon intensity measure $M_\Phi \neq 0$, then there is $\gamma \in (0,\infty)$ and a probability measure $\mathbb{Q}$ on $G(d,k)$ such that*

$$\int_{A(d,k)} f dM_\Phi = \gamma \int_{G(d,k)} \int_{L^\perp} f(L+x)\, \lambda_{L^\perp}(dx)\, \mathbb{Q}(dL) \tag{3}$$

for all $f \geqslant 0$ measurable on $A(d,k)$. Moreover, γ and $\mathbb{Q}$ are unique. ☐

We call γ the **intensity** of Φ and $\mathbb{Q}$ the **directional distribution** of Φ. The naming of γ and $\mathbb{Q}$ becomes clear after a few applications of the characterization (2). Taking $\Theta := M_\Phi$ we have

$$\gamma = \Theta_0(G(d,k)) = \frac{1}{\kappa_{d-k}} \mathbb{E}[\Phi(\mathcal{F}_{B^d} \cap \pi_0^{-1}(G(d,k)))] = \frac{1}{\kappa_{d-k}} \mathbb{E}[\Phi(\mathcal{F}_{B^d})]$$

and thus

$$\mathbb{Q}(A) = \frac{\mathbb{E}[\Phi(\mathcal{F}_{B^d} \cap \pi_0^{-1}(A))]}{\mathbb{E}[\Phi(\mathcal{F}_{B^d})]}.$$

Another interpretation of γ is given by the following. It intuitively says that if we measure k-dimensional slices of a set via the flats of a stationary process of flats, then on average we recover the measure of the set up to a multiplicative factor of γ.

Proposition 2.3. *Let $\Phi = \sum_n \delta_{E_n}$ be a stationary k-flat process in $\mathbb{R}^d$ with intensity γ, then $\sum_n \lambda_{E_n}(A)$ is a random variable for each $A \in \mathcal{B}(\mathbb{R}^d)$, and $\mathbb{E}\sum_n \lambda_{E_n}(A) = \gamma\lambda(A)$.*

Proof. We omit the proof of measurability. By Campbell's theorem and our decomposition (3)

$$\begin{aligned}
\mathbb{E}\sum_n \lambda_{E_n}(A) &= \mathbb{E}\int_{A(d,k)} \lambda_E(A)\, \Phi(dE) \\
&= \int_{A(d,k)} \lambda_E(A)\, M_\Phi(dE) \\
&= \gamma \int_{G(d,k)} \int_{L^\perp} \lambda_{L+x}(A)\, \lambda_{L^\perp}(dx)\, \mathbb{Q}(dL) \\
&= \gamma \int_{G(d,k)} \int_{L^\perp} \lambda_L(A-x)\, \lambda_{L^\perp}(dx)\, \mathbb{Q}(dL) \\
&= \gamma \int_{G(d,k)} \lambda(A)\, \mathbb{Q}(dL) \\
&= \gamma\lambda(A). \quad \square
\end{aligned}$$

As usual, if one assumes that Φ is a stationary Poisson k-flat process, one is able to prove many general facts without assuming any more about the law of the process. We give two examples in the following.

Proposition 2.4. *Let $\Phi = \sum_n \delta_{E_n}$ be a stationary Poisson k-flat process in $\mathbb{R}^d$.*

(a) If $k < d/2$, then a.s. no two k-flats of Φ intersect.

(b) If the directional distribution $\mathbb{Q}$ of Φ has no atoms, then a.s. no two k-flats of Φ are translates of each other.

Proof. We employ the second factorial power $\Phi^{(2)}$ of Φ, for which we know $M_{\Phi^{(2)}} = M_\Phi^2$ because Φ is Poisson. For any $A \in \mathcal{B}(A(d,k)^2)$ we have by Campbell's theorem and (3)

$$\begin{aligned}
\mathbb{E} \sum_{n \neq m} 1_A(E_n, E_m) &= \mathbb{E} \int_{A(d,k)^2} 1_A(E, E')\, \Phi^{(2)}(dE, dE') \\
&= \int_{A(d,k)} \int_{A(d,k)} 1_A(E, E')\, M_\Phi(dE)\, M_\Phi(dE') \\
&= \gamma^2 \int_{G(d,k)} \int_{G(d,k)} \int_{L'^\perp} \int_{L^\perp} 1_A(L + x, L' + x')\, \lambda_{L^\perp}(dx)\, \lambda_{L'^\perp}(dx')\, \mathbb{Q}(dL)\, \mathbb{Q}(dL').
\end{aligned}$$

For (a) we take $A := \{(E, E') \in A(d,k)^2 : E \cap E' \neq \emptyset\}$. Then if $1_A(L+x, L'+x') = 1$, we have $(L+x) \cap (L'+x') \neq \emptyset$, so $x \in (L'+x'-L) \cap L^\perp$. But for every $l'+x' \in L'+x'$ there is exactly one $l \in L$ such that $l'+x'-l \in L^\perp$, namely $l = \mathrm{proj}_L(l'+x')$, in which case $l'+x'-l = \mathrm{proj}_{L^\perp}(l'+x')$. It follows that $(L'+x'-L) \cap L^\perp = \mathrm{proj}_{L^\perp}(L'+x')$ which has dimension at most $\dim(L') = k < d/2 < d-k = \dim(L^\perp)$. Thus the inner integral is 0, so the sum is a.s. equal to 0.

For (b) we take $A := \{(E, E') \in A(d,k)^2 : E_i \cap rB^d \neq 0, i = 1, 2, \pi_0(E) = \pi_0(E')\}$. In this case $1_A(L+x, L'+x') = 1_{|x|,|x'|<r} 1_{L=L'}$. Then

$$\begin{aligned}
\mathbb{E} \sum_{n \neq m} 1_A(E_n, E_m) &= \gamma^2 \kappa_{d-k}^2 r^2 \int_{G(d,k)} \int_{G(d,k)} 1_{L=L'}\, \mathbb{Q}(dL)\, \mathbb{Q}(dL') \\
&= \gamma^2 \kappa_{d-k}^2 r^2 \int_{G(d,k)} \mathbb{Q}(\{L'\})\, \mathbb{Q}(dL') \\
&= 0.
\end{aligned}$$

Thus again the sum is a.s. equal to 0. Sending $r \to \infty$ and applying the monotone convergence theorem then gives the result. □

3. References

[1] R. Schneider and W. Weil. *Stochastic and integral geometry.* Springer Science & Business Media, 2008.