Threshold Dynamics

with Applications to Image Processing

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Multiphase Mean Curvature Flow

Investigate algorithms for simulating the mean curvature flow of networks of interfaces under arbitrary surface tensions.

Setup

A domain D is partitioned into N closed sets Σ_i called *phases*. We want to develop algorithms for simulating the L^2-gradient flow of the energy

$$E(\Sigma_1, ..., \Sigma_N) = \sum_{i,j=1}^{N} \sigma_{i,j} \text{Area}(\partial\Sigma_i \cap \partial\Sigma_j)$$

Here, $\sigma_{i,j}$ are given *surface tensions* that satisfy

- $\sigma_{i,j} = \sigma_{j,i}$
- $\sigma_{i,i} = 0$
- $\sigma_{i,j} > 0 \quad \text{for } i \neq j$

Results/Constraints

1. At "smooth" points, $\partial\Sigma_i \cap \partial\Sigma_j$ moves by mean curvature flow.
2. At triple junctions $p \in \partial\Sigma_i \cap \partial\Sigma_j \cap \partial\Sigma_k$, the *Herring angle condition* holds:

$$\sigma_{i,j} n_{i,j}(p) + \sigma_{j,k} n_{j,k}(p) + \sigma_{k,l} n_{k,l}(p) = 0$$

where $n_{i,j}$ denotes the unit normal to $\partial\Sigma_i \cap \partial\Sigma_j$ (pointing from Σ_i into Σ_j). Equivalently, in terms of the opening angles $\theta_1, \theta_2, \theta_3$,

$$\frac{\sin(\theta_1)}{\sigma_{2,3}} = \frac{\sin(\theta_2)}{\sigma_{1,3}} = \frac{\sin(\theta_3)}{\sigma_{1,2}}$$

MBO Algorithm

Let $G_{\delta t}(x)$ denote the heat kernel

$$G_{\delta t}(x) = \frac{1}{(4\pi\delta t)^{d/2}} e^{-\|x\|^2/4\delta t}$$

Given the partition $\Sigma_1^k, ..., \Sigma_N^k$ at time $t = (\delta t)k$, obtain the partition Σ_i^{k+1} at the next time step as follows:

Step 1. (Convolution)

$$\phi_i^k = G_{\delta t} * \mathbf{1}_{\Sigma_i^k}$$

Step 2. (Thresholding)

$$\Sigma_i^{k+1} = \{x \in D | \phi_i^k(x) > \phi_j^k(x) \text{ for all } j \neq i\}$$

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Problem: This method only ensures the Herring angle condition for surface tensions $\sigma_{i,j} = 1$ for all $i \neq j$.

Approximate Energy

$$Area(\partial\Sigma_i \cap \partial\Sigma_j) \approx \frac{1}{\sqrt{\delta t}} \int \mathbf{1}_i G_{\delta t} * \mathbf{1}_j dx$$

$$\Longrightarrow \; E(\Sigma_1, ..., \Sigma_N) \approx E_{\delta t}(\Sigma_1, ..., \Sigma_N) = \sum_{i,j=1}^{N} \sigma_{i,j} \frac{1}{\sqrt{\delta t}} \int \mathbf{1}_{\Sigma_i} G_{\delta t} * \mathbf{1}_{\Sigma_j} dx$$

More generally, for $\mathbf{u} = (u_1, ..., u_N) : D \to [0,1]^N$ such that $\sum_j u_j \equiv 1$, let

$$E_{\delta t}(\mathbf{u}) = \sum_{i,j=1}^{N} \sigma_{i,j} \frac{1}{\sqrt{\delta t}} \int u_i G_{\delta t} * u_j dx$$

The linearization of $E_{\delta t}$ at $\mathbf{u}^*$ is denoted

$$\mathcal{L}_{E_{\delta t}}(\mathbf{u}^*, \mathbf{u}) = \frac{2}{\sqrt{\delta t}} \sum_{i=1}^{N} \int u_i \left(\sum_{j \neq i} \sigma_{i,j} G_{\delta t} * u_j^* \right) dx$$

EO Algorithm

The EO-Algorithm minimizes the linearization of the approximate energy to obtain a threshold dynamics scheme for multiphase mean curvature motion.
Namely, given the partition $\Sigma_1^k, ..., \Sigma_N^k$ at time $t = (\delta t)k$, obtain the partition at the next time step as follows:

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$$\phi_i^k = \sum_{j=1}^{N} \sigma_{i,j} G_{\delta t} * \mathbf{1}_{\Sigma_j^k}$$

Step 2. (Thresholding)

$$\Sigma_i^{k+1} = \{x \in D | \phi_i^k(x) < \phi_j^k(x) \text{ for all } j \neq i\}$$

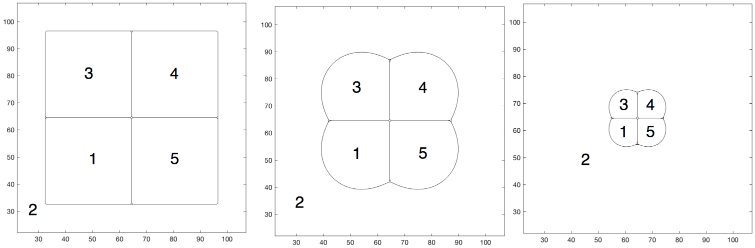
Stability

We say the surface tensions $\sigma_{i,j}$ are *conditionally negative semi-definite* if

$$\sum_{i=1}^{N} \xi_i = 0 \implies \sum_{i,j=1}^{N} \sigma_{i,j}\xi_i\xi_j \leq 0$$

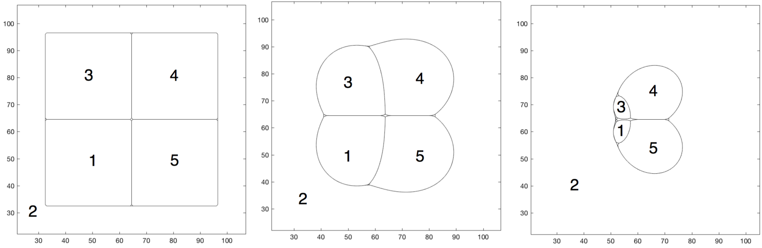
For such surface tensions, the EO-Algorithm is unconditionally gradient stable, i.e. each iteration decreases the approximate energy $E_{\delta t}$.
In applications, many surface tensions are conditionally negative semi-definite (e.g. Read-Shockley type surface tensions).

Demo



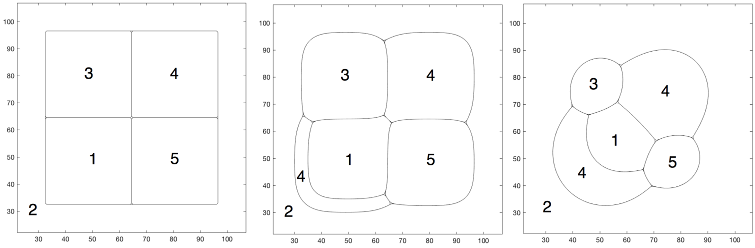
$$(\sigma_{i,j}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Different surface tensions $\sigma_{i,j} \rightsquigarrow$ different dynamics



$$(\sigma_{i,j}) = \begin{pmatrix} 0 & 1.5 & 1.5 & 1 & 1 \\ 1.5 & 0 & 1.5 & 1 & 1 \\ 1.5 & 1.5 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1.5 \\ 1 & 1 & 1 & 1.5 & 0 \end{pmatrix}$$

Nucleation/Wetting



$$(\sigma_{i,j}) \sim \begin{pmatrix} 0 & 1.5 & 1 & 0.5 & 1 \\ 1.5 & 0 & 1 & 0.5 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0.5 & 0.5 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Image Segmentation

- Given an input image, how to determine the foreground vs. background?
- More generally, how to separate an image into n regions?

Mumford-Shah Model

Energy minimization approach:

$$E_{\mathrm{MS}}(u, \Gamma) = \int_{\Omega \setminus \Gamma} \|\nabla u\|^2 \, dx + \mu \cdot \mathrm{Len}(\Gamma) + \lambda \int_{\Omega} (u - f)^2 \, dx$$

where

- $\mu, \lambda > 0$
- $f : \Omega \to \mathbb{R}$ input image
- $\Gamma \subseteq \Omega$ closed set, union of finite number of curves, the boundaries of regions
- u piecewise smooth approximation of f

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Terms penalize for lack of smoothness in regions, length of boundaries of regions, and lack of image fidelity.

Mumford-Shah Model (cont.)

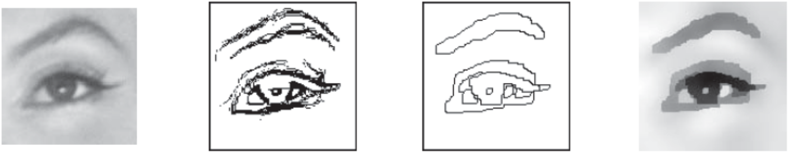

Image credit [4], left to right:

- original
- high contrast areas ($\|\nabla f\|^2$ large)
- Γ boundaries shown
- smooth approximation u shown

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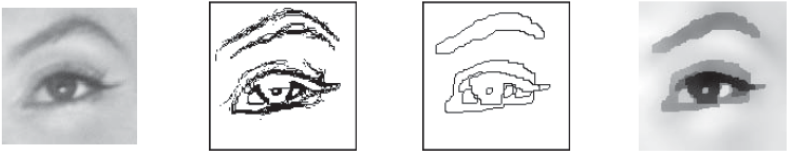

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Works well, but numerically too difficult for practical applications.

Chan-Vese Model

Allow input image $f : \Omega \to \mathbb{R}^d$, e.g. $f = (R, G, B, A)$.
Simplify Mumford-Shah by taking u piecewise constant,

$$E_{\mathrm{CV}}(\Omega_1, \ldots, \Omega_n, C_1, \ldots, C_n) = \sum_{i=1}^{n} \int_{\Omega_i} \|C_i - f\|^2 \, dx + \lambda \cdot \mathrm{Len}(\partial \Omega_i)$$

where

- $\lambda > 0$
- $\Omega_1, \ldots, \Omega_n$ partition Ω
- $u = C_i$ on Ω_i.

Chan-Vese Model (cont.)

Rewrite with $u_i = 1_{\Omega_i}, g_i = \| C_i - f \|^2$,

$$E_{\mathrm{CV}}(\Omega_1, \ldots, \Omega_n, C_1, \ldots, C_n) = \sum_{i=1}^{n} \int_{\Omega} u_i g_i \, dx + \lambda \cdot \mathrm{Len}(\partial \Omega_i)$$

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Best (minimizing) choice for C_i is

$$C_i = \frac{\int_{\Omega} u_i f \, dx}{\int_{\Omega} u_i \, dx}$$

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But still numerically too difficult to find best $\Omega_1, \ldots, \Omega_n$.

Approximate Chan-Vese Model

Two simplifications

- Allow $0 \leq u_i \leq 1$ and $\sum_{i=1}^{n} u_i = 1$ to make problem convex

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- $\mathrm{Len}(\partial\Omega_i) \approx \sum_{j=1, j\neq i}^{n} \sqrt{\frac{\pi}{\delta t}} \int_{\Omega} u_i G_{\delta t} * u_j \, dx$, $G_{\delta t}$ heat kernel
  - $G_{\delta t} = \frac{1}{4\pi\delta t} \exp\left(-\frac{\|x\|^2}{4\delta t}\right)$
  - $\delta t \ll 1$

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Problem becomes minimizing

$$\mathcal{E}^{\delta t}(u_1, \ldots, u_n) = \sum_{i=1}^n \int_\Omega \left(u_i g_i + \lambda \sum_{j=1, j\neq i}^n \sqrt{\frac{\pi}{\delta t}} u_i G_{\delta t} * u_j \right) dx$$

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It turns out minimum will have $u_i \in \{0, 1\}$ for all i.

Approximate Chan-Vese Model Iteration Scheme

Given $u^k = (u_1^k, \ldots, u_n^k)$ some iteration, linearize

$$\mathcal{E}^{\delta t}(u_1, \ldots, u_n) \approx \mathcal{E}^{\delta t}(u_1^k, \ldots, u_n^k) + \mathcal{L}(u_1 - u_1^k, \ldots, u_n - u_n^k, u_1^k, \ldots, u_n^k)$$

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- $g_i^k = \left\| C_i^k - f \right\|^2$
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Minimize $\mathcal{L}$ to determine u^{k+1}.

Approximate Chan-Vese Model Iteration Scheme (cont.)

$$\mathcal{L}(u_1, \ldots, u_n, u_1^k, \ldots, u_n^k) = \sum_{i=1}^{n} \int_\Omega u_i \phi_i^k \, dx$$

Minimum attained at

$$u_i^{k+1} = \begin{cases} 1 & \phi_i^k(x) = \min_j \phi_j^k(x) \\ 0 & \text{else} \end{cases}$$

Minimum can be computed in parallel for each x.
This iteration scheme has decreasing energy.

Threshold Dynamics Formulation

Iteration scheme suggests algorithm

Step 0. Given initial partition $\Omega_1^0, \dots, \Omega_n^0$ of Ω define corresponding $u_i^0 = 1_{\Omega_i^0}$. Pick tolerance $\tau > 0$.

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Step 1. Given $u^k = (u_1^k, \ldots, u_n^k)$, compute C_i^k, g_i^k, ϕ_i^k as before. Use FFT to compute convolutions.

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Step 2. Thresholding: let $\Omega_i^{k+1} = \{x : \phi_i^k(x) < \min_{j \neq i} \phi_j^k(x)\}$ and define corresponding $u_i^{k+1} = 1_{\Omega_i^{k+1}}$.

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Step 3. Compute normalized difference of successive iterations

$$e^{k+1} = \frac{1}{|\Omega|} \int_\Omega \sum_{i=1}^n \left\| u_i^{k+1} - u_i^k \right\|^2 dx$$

If $e^{k+1} \leq \tau$ stop, else goto step 1.

Threshold Dynamics Formulation (cont.)

$d = 1, n = 2, \delta t = 0.03$ and $\lambda = 0.01$. Convergence after 15 iterations in 0.1188 seconds, image credit [1].

Demo

References

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